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THE IMPROVEMENT OF ACCURACY OF AN APPROXIMATE METHOD FOR EVALUATION OF EXPANDED UNCERTAINTY THROUGH MODELLING THE CHANGES OF COVERAGE FACTOR

Taking a simple case of direct measurement with a limited number of standard uncertainty components as an example, the approximated method of evaluation of expanded uncertainty of measurement has been presented. It has been assumed that the knowledge about the convolution of component distributions enables the expanded uncertainty's estimation which might be admitted to be the accurate one. It made possible to carry out the comparative analysis of accuracy of the approximated method of the expanded uncertainty's estimation. Also it allowed to point out the way of improvement in accuracy of this estimation by modelling variations of the coverage factor's values expressed versus variations of the ratio of a component standard uncertainty at the defined most often used values of level of confidence α .

1. INTRODUCTION

The evaluation of expanded uncertainty of a measurement result is always related to the necessity of choosing an appropriate method of evaluation, in accordance with accepted rules [2]. It is most difficult to make a conscious choice, since apart from the knowledge that every method of evaluation of expanded uncertainty is an approximated method, there is impossible to evaluate the degree of approximation because there does not exist a defined method which can be considered accurate. Therefore, if we want to talk about the accuracy of approximated methods of evaluation of expanded uncertainty, it is necessary to define such a method. It will allow to compare the accuracy of certain approximated methods.

The measure of difference between the approximated method and the method assumed to be accurate is defined by error

$$\delta = \frac{|u_{e,app} - u_{e,acc}|}{u_{e,acc}} \cdot 100\% \quad (1)$$

where $u_{e,app}$ is the approximated expanded uncertainty, and $u_{e,acc}$ is the accurate expanded uncertainty.

We know that expanded uncertainty is described by equation

$$u_e = k(\alpha) \cdot u_c \quad (2)$$

The coverage factor $k(\alpha)$ is the standardised variable of the distribution which is a convolution of component distributions. The characteristic parameter of this distribution is the combined standard deviation σ_c equal to the combined standard uncertainty u_c

$$u_c = \sigma_c = \sqrt{\sum_{j=1}^N u_j^2} = \sqrt{\sum_{j=1}^N \sigma_j^2} \quad (3)$$

where: $j = 1, 2, \dots, N$; u — standard uncertainty.

In order to calculate expanded uncertainty with assumed probability α , we should know the distribution which is a convolution of j -th component distributions with standard deviation σ_j . We have assumed that the knowledge of the convolution of component distributions allows to evaluate the expanded uncertainty, and that it may be considered an accurate evaluation [3, 4, 5].

As an example, we will analyse a direct measurement with a limited number of independent component errors. It is assumed that there appear random errors Δ_A with distributions close to a normal distribution, as well as measuring devices errors Δ_B which can be assigned a rectangular distribution. The combined standard uncertainty for this case is equal

$$u_c = \sqrt{u_A^2 + u_B^2} \quad (4)$$

where: u_A and u_B — standard uncertainties of type A and type B .

This is a common situation in measurements. This is also a critical situation in the evaluation of expanded uncertainty, since, because of the small number of component errors, we cannot use the central limit theorem which would allow to accept the assumption about the divergence to normal distribution of an unknown distribution which is a convolution of component distributions.

The knowledge of the convolution of normal distribution and rectangular distribution [1, 3, 4], calculated for various relationships between standard deviations on normal and rectangular distributions, allows to determine the values of coverage factor k , which will be denoted $k_{NR}(\alpha)$. Therefore accurate expanded uncertainty is defined as:

$$u_{e,acc} = k_{NR}(\alpha) \cdot u_c \quad (5)$$

It can be shown that error defined by (1) is equivalent in this case to the error of evaluation of coverage factor $k(\alpha)$

$$\delta_k = \frac{|k(\alpha) - k_{NR}(\alpha)|}{k_{NR}(\alpha)} \cdot 100 [\%] \quad (6)$$

Because the calculation of the convolution of component distributions is highly time-consuming, and because the information about the values of $k_{NR}(\alpha)$ is not widely available, approximated methods are used [2, 3, 4, 5]. The basic problem is the evaluation of coverage factor k , which is the function of probability α and of the ratio of values σ_N/σ_R or the reverse ratio σ_R/σ_N .

2. THE ACCURACY OF EVALUATION OF COVERAGE FACTOR $k(\alpha)$

One of the approximated methods used to evaluate the coverage factor $k(\alpha)$ is the method based on the hypothesis that an unknown convolution of component distributions may be approximated by a distribution of greater standard deviation [3, 4, 5]. This hypothesis is true for the boundary cases

$$\text{if } \frac{\sigma_N}{\sigma_R} \rightarrow \infty \quad \text{than} \quad k(\alpha) \rightarrow k_N(\alpha) = z(\alpha) \quad (7)$$

$$\text{if } \frac{\sigma_R}{\sigma_N} \rightarrow \infty \quad \text{than} \quad k(\alpha) \rightarrow k_R(\alpha) = \sqrt{3} \cdot \alpha \quad (8)$$

where: $k_N(\alpha)$ and $k_R(\alpha)$ are standardised variable of the normal and rectangular distributions, respectively.

However, this hypothesis does not determine the possibility of evaluation when there is an equality relationship between the standard uncertainty, or if this relationship is close to 1. The examination of error of this method, resulting from relation (6) and described by relation (9) and (10) allows to draw some general conclusions about the possibility of such evaluation [3, 4, 5]

$$\delta_1 = \frac{|k_N(\alpha) - k_{NR}(\alpha)|}{k_{NR}(\alpha)} \cdot 100\% \quad (9)$$

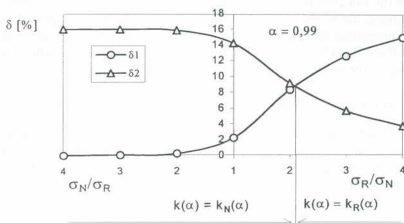
$$\delta_2 = \frac{|k_R(\alpha) - k_{NR}(\alpha)|}{k_{NR}(\alpha)} \cdot 100\% \quad (10)$$

Figure 1 shows, for a chosen value of probability $\alpha = 0,99$, the changes of calculated errors in the function of the ratio σ_N/σ_R and σ_R/σ_N .

The conclusions from Fig. 1 may be generalised for all probabilities α in the range of values $0,9 \leq \alpha \leq 0,9973$, assuming that:

- * for the ratio $\frac{\sigma_N}{\sigma_R} \geq 1$ and for $\frac{\sigma_R}{\sigma_N} < 2$, the normal distribution, for which $k(\alpha) = k_N(\alpha)$, is a better approximation of the convolution of distributions $N * R$.
- * for the ratio $\frac{\sigma_R}{\sigma_N} > 2$ the rectangular distribution, for which $k(\alpha) = k_R(\alpha)$ is a better approximation of the convolution of distributions $N * R$.

The error of this evaluation of $k(\alpha)$ does not exceed 23%, if the above relations are fulfilled. The error is related to the value of probability α . The greater the value of α , the greater is the error.

Fig. 1. Errors of approximated method for $\alpha = 0.99$

3. IMPROVING THE ACCURACY OF THE EVALUATION OF COVERAGE FACTOR $k(x)$

In order to improve the accuracy of approximate evaluation of $k(x)$ we have tried to make a mathematical model of the curve convergent to the curve representing the changes of values of $k_{NR}(x)$. In order to determine the range of modelling we have examined the convergence of $k_{NR}(x)$ to $k_N(x)$ in the range for which $\sigma_N > \sigma_R$, and of $k_{NR}(x)$ to $k_R(x)$ in the range for which $\sigma_R > \sigma_N$, for different, most widely used values of probability α .

It can be shown that the convergence of $k_{NR}(x)$ to $k_N(x)$ for $\sigma_N > \sigma_R$ and of $k_{NR}(x)$ to $k_R(x)$ for $\sigma_R > \sigma_N$ is different for different probability α .

Fig. 2 shows an example of examining the convergence of $k_{NR}(x)$ for $\alpha = 0.95$, assuming that the values of coverage factor $k_N(0.95)$ and $k_R(0.95)$ are within the range determined by the error $\pm \Delta$, whose relative value is $\pm 1\%$.

- * When $\sigma_N \geq \sigma_R$, $k_{NR}(0.95)$ is convergent to $k_N(0.95)$ already for the ratio $\sigma_N/\sigma_R \geq 1.5$
- * When $\sigma_R > \sigma_N$, $k_{NR}(0.95)$ is convergent to $k_R(0.95)$ already for the ratio $\sigma_R/\sigma_N > 7$.

In the range where $\sigma_N \geq \sigma_R$, the value of $k_{NR}(x)$ relatively quickly achieves the value convergent to the value of the standardised variable of normal distribution $k_N(x)$. It may be shown that for all considered probabilities α (Fig. 2) $k_{NR}(x)$ becomes convergent to $k_N(x)$ when $\sigma_N/\sigma_R \geq 2$.

In the range where $\sigma_R \geq \sigma_N$ we can see a strong influence of normal distribution on the convolution of distributions $N * R$. The values of $k_{NR}(x)$ converges to the standardised variable of rectangular distribution much slower. For $\alpha = 0.99$ $k_{NR}(x)$

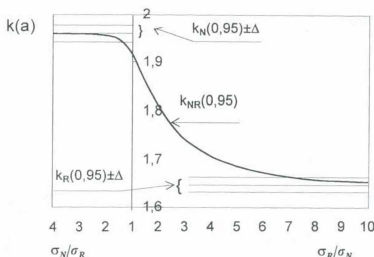


Fig. 2. Illustration of convergence of the factor $k_{NR}(0,95)$ to the factor $k_N(0,95)$ where $\sigma_N \geq \sigma_R$ and to the factor $k_R(0,95)$ where $\sigma_R > \sigma_N$, with the assumed accuracy of estimation of convergence defined by an error equal to $\pm 1\%$

achieves the convergence to $k_R(x)$ only for the ratio $\sigma_R/\sigma_N > 20$ and for $\alpha = 0.9973$ it is respectively $\sigma_R/\sigma_N = 50$.

We have assumed the following modelling range:

$$1 \leq \frac{\sigma_R}{\sigma_N} \leq 10 \quad \text{and} \quad 1 \leq \frac{\sigma_N}{\sigma_R} \leq 2 \quad (11)$$

The reason for this choice is as follows: for a tenfold difference between the values of component uncertainties, one of them becomes the dominant uncertainty which means that the remaining uncertainty is negligible. Moreover, for $\sigma_N/\sigma_R \geq 2$ the curve describing the changes of $k_{NR}(x)$ has a constant value, convergent to a constant value of $k_N(x)$.

From among a number of possible solutions we suggest the equation describing a model of coverage factor $k_m(x)$, convergent to $k_{NR}(x)$ in the range $0.5 \leq \sigma_R/\sigma_N \leq 10$.

$$k_m(x) = A \cdot \exp \left[B \cdot \left(\frac{\sigma_R}{\sigma_N} \right)^C \right] \quad (12)$$

Figure 3 shows the curves of changes of $k_m(x)$, calculated from (12) for selected, most often used probabilities α , and respective curves of $k_{NR}(x)$.

Table I comprises the values of A ; B and C for all considered probabilities α and the values of modelling errors δ_k , described by relation (13).

$$\delta_k = \frac{|k_m(x) - k_{NR}(x)|}{k_{NR}(x)} \cdot 100\% \quad (13)$$

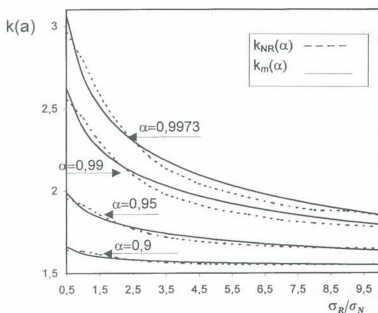


Fig. 3. Coverage factors $k_{NR}(\alpha)$ and their models $k_m(\alpha)$ versus ratio of the standard deviation of normal and rectangular distribution, for chosen values of α

Table 1. Values of modelling constants and modelling errors

$k_m(\alpha) = A \cdot \exp \left[B \cdot \left(\frac{\sigma_R}{\sigma_N} \right)^C \right]$				
α	A	B	C	δ_k [%]
0.9	1.52	0.064	0.5	1.1
0.95	0.94	0.7	0.1	1.6
0.9545	1.27	0.41	0.18	2.0
0.99	0.44	1.69	0.08	2.9
0.9973	0.245	2.4	0.074	3.3

Table 1 shows that δ_k does not exceed 3.3% in the whole modelling range considered, and it grows with the growing value of α .

We can obtain greater accuracy when we model the curve of changes of $k_{NR}(\alpha)$ separately for $\sigma_N/\sigma_R > 1$ and $\sigma_R/\sigma_N \geq 1$.

- * For $\sigma_R/\sigma_N \geq 1$ the equation (12) is a very good description of the model of $k_{NR}(\alpha)$.
- * For $\sigma_N/\sigma_R \geq 1$ $k_m(\alpha)$ may be described by equation (14).

$$k_m(\alpha) = A \cdot \left(\frac{\sigma_N}{\sigma_R} \right)^D \cdot \ln \left[B \cdot \left(\frac{\sigma_N}{\sigma_R} \right)^C \right] \quad (14)$$

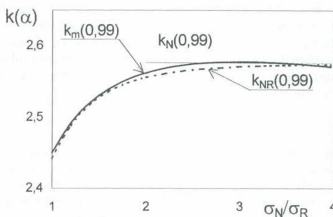


Fig. 4. Coverage factor $k_{NR}(0.99)$ and its model $k_m(0.99)$ versus the ratio of standard uncertainties σ_N/σ_R for $\alpha=0.99$

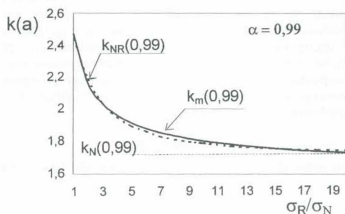


Fig. 5. Coverage factor $k_{NR}(0.99)$ versus the ratio of standard deviations σ_R/σ_N , for $\alpha=0.99$

T a b l e 2. Values of modelling constants and modelling errors

$u_A \geq u_B$						$u_B \geq u_A$				
$k(x) = A \left(\frac{\sigma_N}{\sigma_R} \right)^D \cdot \ln \left[B \cdot \left(\frac{\sigma_N}{\sigma_R} \right)^C \right]$						$k(x) = A \cdot \exp \left[B \cdot \left(\frac{\sigma_R}{\sigma_N} \right)^C \right]$				
α	A	B	C	D	δ_k [%]	α	A	B	C	δ_k [%]
0.9	1.43	3.16	-0.12	0.11	0.5	0.9	0.51	0.078	-0.5	0.6
0.95	1.8	2.9	0.25	-0.18	0.2	0.95	1.52	0.24	-0.5	0.8
0.9545	1.9	2.8	0.24	-0.18	0.6	0.9545	1.525	0.25	-0.5	0.5
0.99	2.23	3.0	0.4	-0.26	0.2	0.99	1.55	0.467	-0.5	1.4
0.9973	2.92	2.6	0.4	-0.28	0.4	0.9973	1.36	0.73	-0.37	1.5

Fig. 4 shows an example of changes of $k_{NR}(\alpha)$ and $k_m(\alpha)$, calculated from (14), for a selected value of probability $\alpha = 0.99$, for the case when $\sigma_N \geq \sigma_R$.

Fig. 5 shows $k_m(\alpha)$ calculated from (12) for the case, when $\sigma_R \geq \sigma_N$.

Table 3 comprises the values of A ; B ; C ; D , for all considered probabilities α and for errors δ_k . The modelling error δ_k , calculated from (14) does not exceed 1.5% in the whole modelling range examined.

4. CONCLUSIONS

The problem of accuracy of the evaluation of expanded uncertainty of measurement results always raises question, since it is difficult to decide if a 20% error is big or small in view of many other approximations made for evaluation of expanded uncertainty. It seems, however, that from the point of view of metrology it does make a difference whether it is a 2%, 20% or 40% error. Since in the evaluation of uncertainty it is the researcher who makes the decisions regarding both the values of some parameters and the using of particular procedures, it is imperative that we should be aware of the results of his decisions. Therefore it seems sensible to establish the evaluation of expanded uncertainty, which can be considered accurate, and to publish the results of comparative analysis of accuracy of approximate evaluation methods used.

The model descriptions presented have reduced considerably the approximation error of evaluation of expanded uncertainty for a common measurement situation. The model descriptions included in Tables 1 and 2 may replace a comprehensive table of values of coverage factor $k_{NR}(\alpha)$ for the convolution of normal distribution and rectangular distribution.

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POPRAWA DOKŁADNOŚCI PRZYBLIŻONEJ METODY OCENY
NIEPEWNOŚCI CAŁKOWITEJ PRZEZ MODELOWANIE ZMIAN
WSPÓŁCZYNNIKA ROZSZERZENIA

S t r e s z c z e n i e

Wykazano, że dokładność oceny współczynnika rozszerzenia, który jest zmienną standaryzowaną rozkładu będącego spletem rozkładów składowych, decyduje o dokładności oceny niepewności całkowitej. W celu zwiększenia dokładności przybliżonej oceny współczynnika rozszerzenia podjęto próby utworzenia modelu matematycznego krzywej zmian wartości współczynnika rozszerzenia, dla spletu rozkładu normalnego i jednostajnego.