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CONTRIBUTION OF JITTER TO THE ERROR OF AMPLITUDE ESTIMATION OF A SINUSOIDAL SIGNAL

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Abstract

Jitter and phase noise are non-ideal effects that lead to uncertainties in estimating the parameters of a sinusoidal signal. In this paper, the particular case of the bias induced by these effects on the amplitude estimation is considered. An analytical expression is derived for the relative error of amplitude estimation as a function of number of samples and phase noise standard deviation. It is demonstrated that, in the case of coherent sampling, the relative error is independent of sinusoidal amplitude and offset.

Keywords: analogue-to-digital conversion, jitter, phase noise, uncertainty, sinusoidal signal.

1. Introduction

Increasing sampling rates in today's data acquisition systems leads to more stringent requirements in terms of analogue-to-digital converters and sampling circuits. One of those requirements, studied in this paper, has to do with jitter and phase noise.

A typical measurement technique consists in applying a sinusoidal signal to a circuit and measuring the parameters of the signal at a given point of that circuit in order to estimate something about that circuit. Examples of measurement systems that use this technique, and in particular are based on estimating the amplitude of an acquired sine wave, are, just to name a few:

– Impedance measurement – A sinusoidal voltage is applied to the series of an unknown impedance and a sampling resistor with known value. The voltage on that resistance is then measured, together with the voltage on the unknown impedance, in order to estimate the value of the current going through the unknown impedance. The module of the impedance is given by the ratio of the two sinusoidal amplitudes [1].
– Non-destructive testing using eddy currents – A sinusoidal current is applied to a coil in order to produce a sinusoidal magnetic field. That coil is placed close to the surface of the material under test in order to induce eddy currents in it. Any defect in the material under test will cause a non-uniform eddy current being produced. That current is then determined indirectly by measuring its magnetic field with a second coil or giant magneto-resistive sensor (GMR). By making the probe scan the surface while estimating, for each point, the amplitude of the magnetic field induced by the eddy currents it is possible to draw a 2D map the induced currents and from that to detect and characterize defects [2-4].
– Strain measurement with strain gauges – One or more strain gauges are typically connected in a Wheatstone bridge configuration to estimate the strain to which a given surface is subjected to. In applications where noise is a problem, the bridge is powered with a sinusoidal voltage. The unbalance voltage, which is also sinusoidal, will have an amplitude proportional to the strain to be measured.

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These are some of the examples of the many that could be given, from analogue-to-digital (ADC) converter testing [5-9] to particle size and velocity determination using laser anemometry [10], that justify the importance of sinusoidal amplitude estimation in today's world dominated by digital systems.

There are many sources of uncertainty that affect a data acquisition system, namely, additive noise, phase noise, jitter, quantization error, frequency error, harmonic distortion, non-linearity [11], among others. All these sources contribute to the uncertainty of measurements made with digitized data. In [12] it was shown that the presence of additive noise leads to a bias on sinusoidal amplitude estimation using least-mean-squares error sine-fitting. In [13], an analytical expression for the standard deviation of amplitude estimation as a function of harmonic distortion, number of samples and phase noise/jitter standard deviation, has been derived. In [14] the bias on sinusoidal amplitude estimation was computed from the standard deviation of additive noise and the standard deviation of phase noise/jitter, however it is only valid for small amounts of phase noise and jitter.

Here we present the derivation of an analytical expression which only takes into account the effect of phase noise and jitter, but has a wider range of use. It can always be combined with the expression derived in [14] to have an account of both additive noise and phase noise/jitter which usually are present in practical conditions.

In [15] the IEEE 1057 standard method [6], which minimizes the square of the residuals, was studied. It is pointed out that a bias in the estimated amplitude arises due to jitter at the sampling instant. It is also shown how to compute that bias in the asymptotic case (infinite number of samples). In the limit when the number of samples goes to infinity, the expression presented here tends to the expression given in [15] as will be demonstrated.

2. Least Mean Square Error Sinusoidal Estimation

Consider M data points $z_1, z_2, \ldots, z_M$ given by:

$$z_i = C + A\cos\left[\omega_x\left(t_i + \delta_i\right) + \varphi\right] \quad \text{with} \quad i = 1,\ldots,M, \tag{1}$$

where C is the offset, A is the amplitude, φ is the initial phase, ω_x is the angular frequency t_i are the sampling instants and δ_i is the sampling instant jitter. Generally the initial phase of the sine wave is not controlled and thus varies from acquisition to acquisition and from measurement to measurement. Statistically we can consider it to be a random variable uniformly distributed in an interval of length 2π. In this work we consider only the presence of normally distributed jitter at the sampling instants and represent it by a null mean random variable δ_i with standard deviation σ_t. To ease the derivations that follow, we will introduce the random variable $\theta_i = \omega_x \delta_i$. This variable will be a null mean random variable with standard deviation $\sigma_\theta = \omega_x \sigma_t$ since it is just a constant (ω_x) times a normally distributed random variable (δ_i) with standard deviation σ_t. Equation (1) can thus be written as:

$$z_i = C + A\cos\left(\omega_x t_i + \theta_i + \varphi\right) . \tag{2}$$

We wish to estimate the sine wave that best fits, in a least square error sense, to these M points. The estimates of the sine wave are obtained, in a matrix form, with [1]:

$$\begin{bmatrix} \widehat{A_I} \\ \widehat{A_Q} \\ \widehat{C} \end{bmatrix} = \left(D^T D\right)^{-1} D^T \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_M \end{bmatrix} \text{ with } D = \begin{bmatrix} \cos(\omega_a t_1) & \sin(\omega_a t_1) & 1 \\ \cos(\omega_a t_2) & \sin(\omega_a t_2) & 1 \\ \cdots & \cdots & \cdots \\ \cos(\omega_a t_M) & \sin(\omega_a t_M) & 1 \end{bmatrix} \text{ and } \widehat{A} = \sqrt{\widehat{A_I}^2 + \widehat{A_Q}^2}, \tag{3}$$

where ω_a is the angular frequency of the sinusoid we are trying to adjust to the data. Here we will assume that the frequency of the signal is exactly known and its value is used to fit the sine wave ($\omega_a = \omega_x$). If the frequency is unknown, a four-parameter sine-fitting algorithm can be used to estimate the frequency [6]. This is an iterative procedure that approximates in each step the frequency of the sine wave that best fits the data. It terminates when the frequency change, from the previous step, is smaller than a chosen bound. There will necessarily be a frequency error which will have some impact on the estimation of the other three parameters. This, however, is generally considered to be negligible. In the future, further work is required to compute the actual effect of frequency error on amplitude estimation in order to gauge its importance.

We will also assume that the number of acquired samples (M) covers exactly an integer number of periods (J) of the sine wave we are trying to fit to the data. Consequently, from (3):

$$\widehat{A}^2 = \widehat{A_I}^2 + \widehat{A_Q}^2 = \frac{4}{M^2} \sum_{i,j} z_i z_j \cos\left[\omega_a\left(t_i - t_j\right)\right]. \tag{4}$$

From now on, for the sake of compactness, we will eliminate the summation limits and assume that all indices go from 1 to M. The summation in (4) is thus a double summation on i and j which go from 1 to M.

3. Mean of Square Estimated Amplitude

The expected value of the square of the estimated sine wave amplitude is, from (4):

$$E\left\{\widehat{A}^2\right\} = \frac{4}{M^2} \sum_{i,j} E\left\{z_i z_j\right\} \cos\left[\omega_a\left(t_i - t_j\right)\right]. \tag{5}$$

Note that the expected value of a sum is equal to the sum of the expected values and that the expected value of a random variable times a constant is equal to that constant times the expected value of the random variable.

Using (2) we can write:

$$E\left\{z_i z_j\right\} = E\left\{\left[C + A\cos\left(\omega_x t_i + \theta_i + \varphi\right)\right] \cdot \left[C + A\cos\left(\omega_x t_j + \theta_j + \varphi\right)\right]\right\}, \tag{6}$$

which can be written as:

$$\begin{aligned} E\left\{z_i z_j\right\} &= C^2 + A^2 E\left\{\cos\left(\omega_x t_i + \varphi + \theta_i\right)\cos\left(\omega_x t_j + \varphi + \theta_j\right)\right\} + CAE\left\{\cos\left(\omega_x t_i + \varphi + \theta_i\right)\right\} + CAE\left\{\cos\left(\omega_x t_j + \varphi + \theta_j\right)\right\} = \\ &= C^2 + \frac{1}{2} A^2 E\left\{\cos\left(\omega_x t_i + \omega_x t_j + 2\varphi + \theta_i + \theta_j\right)\right\} + \frac{1}{2} A^2 E\left\{\cos\left(\omega_x t_i - \omega_x t_j + \theta_i - \theta_j\right)\right\} + \\ &+ CAE\left\{\cos\left(\omega_x t_i + \varphi + \theta_i\right)\right\} + CAE\left\{\cos\left(\omega_x t_j + \varphi + \theta_j\right)\right\}. \end{aligned} \tag{7}$$

Considering that φ is a uniformly distributed random variable between 0 and 2π, we have:

$$E\left\{z_i z_j\right\} = C^2 + \frac{1}{2}A^2 E\left\{\cos\left(\omega_x t_i - \omega_x t_j + \theta_i - \theta_j\right)\right\}, \tag{8}$$

since:

$$E\left\{\cos\left(a+\varphi\right)\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left(a+\varphi\right)d\varphi = 0. \tag{9}$$

To compute the expected value in (8) we have to consider two cases – equal or different values of indices i and j. If they are equal then θ_i and θ_j cancel each other and we cease to have any random variables in the equation. The expected value is thus:

$$E\left\{z_i z_j\right\}\underset{i=j}{=} C^2 + \frac{1}{2}A^2. \tag{10}$$

On the other hand, if the indices are different, we have, considering that θ_i and θ_j are normally distributed random variables with standard deviation σ_θ:

$$E\left\{z_i z_j\right\}\underset{i\neq j}{=} C^2 + \frac{1}{2}A^2\cos\left(\omega_x t_i - \omega_x t_j\right)e^{-\sigma_\theta^2}, \tag{11}$$

since:

$$E\left\{\cos\left(a+\theta\right)\right\} = \int_{-\infty}^{\infty}\cos\left(a+\theta\right)\frac{1}{\sqrt{2\pi\sigma_\theta}}e^{-\frac{\theta^2}{2\sigma_\theta^2}}d\theta = \cos\left(a\right)e^{-\sigma_\theta^2}. \tag{12}$$

Having determined $E\left\{z_i z_j\right\}$ we are now ready to address the determination of $E\left\{\hat{A}^2\right\}$ given by (5). Notice however that the expression to use for the argument of the double summation is different whether indices i and j are equal or not, namely (10) and (11) respectively. In order to proceed with the derivation we need to have complete summations, that is, summations whose indices span all possible values and that have in its argument a single expression for all cases of the indices. The summation in (5) can be divided into two terms: the first one a double summation on i and j for $i \neq j$ using (11) in its argument; and the second one, a simple summation on i where $j = i$ using (10) in its argument. The new double summation, however can be written as a double summation for all values of i and j (using (11)) minus a simple summation with $j = i$ using (11)in its argument. This leads to:

$$E\left\{\hat{A}^2\right\} = \frac{2}{M^2}\sum_{i,j}A^2\cos\left(\omega_x t_i - \omega_x t_j\right)\cos\left[\omega_a\left(t_i - t_j\right)\right]e^{-\sigma_\theta^2} - \frac{2}{M}A^2 e^{-\sigma_\theta^2} + \frac{2}{M}A^2. \tag{13}$$

Note that the terms in C^2 become null because an integer number of periods is acquired. Considering that we know the signal frequency and use it for the sine wave we are trying to fit to the data, ($\omega_a = \omega_x$), we have:

$$E\left\{\hat{A}^2\right\} = A^2 e^{-\sigma_\theta^2} + \frac{1}{M^2}A^2 e^{-\sigma_\theta^2}\sum_{i,j}\cos\left(2\omega_x t_i - 2\omega_x t_j\right) + \frac{2}{M}A^2\left(1 - e^{-\sigma_\theta^2}\right), \tag{14}$$

where we used a trigonometric relation to transform the product of two cosine functions into the sum of two cosine functions. Since we are considering that the sine wave fit to the data covers an integer number of periods, the summation in i and j is 0, leading to:

$$\mu_{\hat{A}^2} = E\left\{\hat{A}^2\right\} = A^2 e^{-\sigma_\theta^2} + \frac{2}{M} A^2 \left(1 - e^{-\sigma_\theta^2}\right). \tag{15}$$

4. Variance of Estimated Square Amplitude

The variance of a random variable can be expressed as the difference between the second moment and the square of the mean. In the case of the variance of the squared estimated amplitude this leads to:

$$\sigma_{\hat{A}^2}^2 = E\left\{\hat{A}^4\right\} - E^2\left\{\hat{A}^2\right\}. \tag{16}$$

Using (4) it is possible to write:

$$\hat{A}^4 = \frac{16}{M^4} \sum_{i,j,k,l} z_i z_j z_k z_l \cos\left[\omega_a\left(t_i - t_j\right)\right] \cos\left[\omega_a\left(t_k - t_l\right)\right]. \tag{17}$$

The expected value of the fourth power of the estimated amplitude is thus:

$$E\left\{\hat{A}^4\right\} = \frac{16}{M^4} \sum_{i,j,k,l} E\left\{z_i z_j z_k z_l\right\} \cos\left[\omega_a\left(t_i - t_j\right)\right] \cos\left[\omega_a\left(t_k - t_l\right)\right]. \tag{18}$$

Inserting (5) and (18) into (16) and making use of $Cov\{x,y\} = E\{xy\} - E\{x\}E\{y\}$, we have for the variance of the square estimated amplitude:

$$\sigma_{\hat{A}^2}^2 = \frac{16}{M^4} \sum_{i,j,k,l} Cov\left\{z_i z_j, z_k z_l\right\} \cos\left[\omega_a\left(t_i - t_j\right)\right] \cos\left[\omega_a\left(t_k - t_l\right)\right]. \tag{19}$$

The actual voltage of a sample, z, can be expressed as $z_i = C + w_i$, where:

$$w_i = A\cos\left(\omega_x t_i + \varphi + \theta_i\right), \tag{20}$$

It can be shown that the variance of the estimated square value of amplitude does not depend on the stimulus signal offset. As such (19) can be written as:

$$\sigma_{\hat{A}^2}^2 = \frac{16}{M^4} \sum_{i,j,k,l} Cov\left\{w_i w_j, w_k w_l\right\} \cos\left[\omega_a\left(t_i - t_j\right)\right] \cos\left[\omega_a\left(t_k - t_l\right)\right]. \tag{21}$$

Since the arguments of the two cosine functions are not random variables we can place them inside the covariance:

$$\sigma_{\hat{A}^2}^2 = \frac{16}{M^4} \sum_{i,j,k,l} Cov\left\{w_i w_j \cos\left[\omega_a\left(t_i - t_j\right)\right], w_k w_l \cos\left[\omega_a\left(t_k - t_l\right)\right]\right\}. \tag{22}$$

Inserting (20) into (22) leads to:

$$\sigma_{\hat{A}^2}^2 = \frac{16A^4}{M^4} \sum_{i,j,k,l} Cov\left\{\begin{matrix}\cos\left(\omega_x t_i + \theta_i + \varphi\right)\cos\left(\omega_x t_j + \theta_j + \varphi\right)\cos\left[\omega_a\left(t_i - t_j\right)\right], \\ \cos\left(\omega_x t_k + \theta_k + \varphi\right)\cos\left(\omega_x t_l + \theta_l + \varphi\right)\cos\left[\omega_a\left(t_k - t_l\right)\right]\end{matrix}\right\}. \tag{23}$$

Again using:

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b), \tag{24}$$

we can write:

$$\sigma_{\hat{A}^2}^2 = \frac{4A^4}{M^4} \sum_{i,j,k,l} Cov \left\{ \begin{array}{l} \cos\left(\omega_x t_i - \omega_x t_j + \theta_i - \theta_j\right)\cos\left[\omega_a\left(t_i - t_j\right)\right] + \\ +\cos\left(\omega_x t_i + \omega_x t_j + \theta_i + \theta_j + 2\varphi\right)\cos\left[\omega_a\left(t_i - t_j\right)\right], \\ \cos\left(\omega_x t_k - \omega_x t_l + \theta_k - \theta_l\right)\cos\left[\omega_a\left(t_k - t_l\right)\right] + \\ +\cos\left(\omega_x t_k + \omega_x t_l + \theta_k + \theta_l + 2\varphi\right)\cos\left[\omega_a\left(t_k - t_l\right)\right] \end{array} \right\}. \tag{25}$$

Now using $Cov\{a+b,c+d\} = Cov\{a,c\} + Cov\{b,d\}$ we can write (25) as:

$$\begin{aligned} \sigma_{\hat{A}^2}^2 = {} & \frac{4A^4}{M^4} \sum_{i,j,k,l} Cov \left\{ \begin{array}{l} \cos\left(\omega_x t_i - \omega_x t_j + \theta_i - \theta_j\right)\cos\left[\omega_a\left(t_i - t_j\right)\right], \\ \cos\left(\omega_x t_k - \omega_x t_l + \theta_k - \theta_l\right)\cos\left[\omega_a\left(t_k - t_l\right)\right] \end{array} \right\} + \\ & + \frac{4A^4}{M^4} \sum_{i,j,k,l} Cov \left\{ \begin{array}{l} \cos\left(\omega_x t_i + \omega_x t_j + \theta_i + \theta_j + 2\varphi\right)\cos\left[\omega_a\left(t_i - t_j\right)\right], \\ \cos\left(\omega_x t_k + \omega_x t_l + \theta_k + \theta_l + 2\varphi\right)\cos\left[\omega_a\left(t_k - t_l\right)\right] \end{array} \right\}. \end{aligned} \tag{26}$$

Using (24) leads to:

$$\begin{aligned} \sigma_{\hat{A}^2}^2 = {} & \frac{A^4}{M^4} \sum_{i,j,k,l} Cov \left\{ \begin{array}{l} \cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right) + \cos\left(\theta_i - \theta_j\right), \\ \cos\left(2\omega_x t_k - 2\omega_x t_l + \theta_k - \theta_l\right) + \cos\left(\theta_k - \theta_l\right) \end{array} \right\} + \\ & + \frac{A^4}{M^4} \sum_{i,j,k,l} Cov \left\{ \begin{array}{l} \cos\left(2\omega_x t_i + \theta_i + \theta_j + 2\varphi\right) + \cos\left(2\omega_x t_j + \theta_i + \theta_j + 2\varphi\right), \\ \cos\left(2\omega_x t_k + \theta_k + \theta_l + 2\varphi\right) + \cos\left(2\omega_x t_l + \theta_k + \theta_l + 2\varphi\right) \end{array} \right\}, \end{aligned} \tag{27}$$

Note that the index of the summations can be exchanged. For example *i* can become *j* and *j* can become *i* without changing the result of the summation. Using this allows us to write (27) as:

$$\begin{aligned} \frac{M^4}{A^4}\sigma_{\hat{A}^2}^2 = {} & \sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right), \cos\left(2\omega_x t_k - 2\omega_x t_l + \theta_k - \theta_l\right)\right\} + \\ & + 2\sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right), \cos\left(\theta_k - \theta_l\right)\right\} + \\ & + \sum_{i,j,k,l} Cov\left\{\cos\left(\theta_i - \theta_j\right), \cos\left(\theta_k - \theta_l\right)\right\} + \\ & + 4\sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i + \theta_i + \theta_j + 2\varphi\right), \cos\left(2\omega_x t_k + \theta_k + \theta_l + 2\varphi\right)\right\}. \end{aligned} \tag{28}$$

Note that, being φ an uniformly distributed random variable between $-\pi$ and π, $E\{\cos(\alpha+\varphi)\} = 0$. Using this we can simplify the 4$^{\text{th}}$ term of the second member of (28) since:

$$\begin{aligned} & Cov\left\{\cos(\alpha+2\varphi), \cos(\beta+2\varphi)\right\} = \\ & = E\left\{\cos(\alpha+2\varphi)\cos(\beta+2\varphi)\right\} - E\left\{\cos(\alpha+2\varphi)\right\}E\left\{\cos(\beta+2\varphi)\right\} = \\ & = \frac{1}{2}E\left\{\cos(\alpha-\beta)\right\} + \frac{1}{2}E\left\{\cos(\alpha+\beta+4\varphi)\right\} - E\left\{\cos(\alpha+2\varphi)\right\}E\left\{\cos(\beta+2\varphi)\right\} = \\ & = \frac{1}{2}E\left\{\cos(\alpha-\beta)\right\}. \end{aligned} \tag{29}$$

We have then:

$$\begin{aligned}\frac{M^4}{A^4}\sigma_{\hat{A}^2}^2 = &\sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right), \cos\left(2\omega_x t_k - 2\omega_x t_l + \theta_k - \theta_l\right)\right\} + \\ &+ 2\sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right), \cos\left(\theta_k - \theta_l\right)\right\} + \\ &+ \sum_{i,j,k,l} Cov\left\{\cos\left(\theta_i - \theta_j\right), \cos\left(\theta_k - \theta_l\right)\right\} + \\ &+ 2\sum_{i,j,k,l} E\left\{\cos\left(2\omega_x\left(t_i - t_k\right) + \theta_i + \theta_j - \theta_k - \theta_l\right)\right\}.\end{aligned} \quad (30)$$

Substituting the covariance by expected values $\left(Cov\{a,b\} = E\{ab\} - E\{a\}E\{b\}\right)$, (30) can be written as:

$$\begin{aligned}\frac{M^4}{A^4}\sigma_{\hat{A}^2}^2 = &\sum_{i,j,k,l} E\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right)\cos\left(2\omega_x t_k - 2\omega_x t_l + \theta_k - \theta_l\right)\right\} - \left[\sum_{i,j} E\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right)\right\}\right]^2 + \\ &+ 2\sum_{i,j,k,l} E\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right)\cos\left(\theta_k - \theta_l\right)\right\} - 2\sum_{i,j} E\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right)\right\}\sum_{k,l} E\left\{\cos\left(\theta_k - \theta_l\right)\right\} + \\ &+ \sum_{i,j,k,l} E\left\{\cos\left(\theta_i - \theta_j\right)\cos\left(\theta_k - \theta_l\right)\right\} - \left[\sum_{i,j} E\left\{\cos\left(\theta_i - \theta_j\right)\right\}\right]^2 + 2\sum_{i,j,k,l} E\left\{\cos\left(2\omega_x\left(t_i - t_k\right) + \theta_i + \theta_j - \theta_k - \theta_l\right)\right\}.\end{aligned} \quad (31)$$

Note that the product of cosine functions may be written as the sum of cosine functions. For instance, looking at the 5th term in the second member of (31), we have:

$$\sum_{i,j,k,l} E\left\{\cos\left(\theta_i - \theta_j\right)\cos\left(\theta_k - \theta_l\right)\right\} = \sum_{i,j,k,l} E\left\{\frac{1}{2}\cos\left(\theta_i - \theta_j + \theta_k - \theta_l\right)\right\} + \sum_{i,j,k,l} E\left\{\frac{1}{2}\cos\left(\theta_i - \theta_j - \theta_k + \theta_l\right)\right\}. \quad (32)$$

Since the cosine functions are inside a summation, we can swap index k with index l in the last term of (32) without altering the summation. Doing this, results in the two terms in the second member of (32) being exactly the same. We thus have:

$$\sum_{i,j,k,l} E\left\{\cos\left(\theta_i - \theta_j\right)\cos\left(\theta_k - \theta_l\right)\right\} = \sum_{i,j,k,l} E\left\{\cos\left(\theta_i - \theta_j + \theta_k - \theta_l\right)\right\}. \quad (33)$$

Applying this reasoning also to the 1st and 3rd terms of the second member of (31) leads to:

$$\begin{aligned}\frac{M^4}{A^4}\sigma_{\hat{A}^2}^2 = &\sum_{i,j,k,l} E\left\{\cos\left[2\omega_x\left(t_i - t_j + t_k - t_l\right) + \theta_i - \theta_j + \theta_k - \theta_l\right]\right\} - \left[\sum_{i,j} E\left\{\cos\left[2\omega_x\left(t_i - t_j\right) + \theta_i - \theta_j\right]\right\}\right]^2 + \\ &+ 4\sum_{i,j,k,l} E\left\{\cos\left[2\omega_x\left(t_i - t_j\right) + \theta_i - \theta_j + \theta_k - \theta_l\right]\right\} - 2\sum_{i,j} E\left\{\cos\left[2\omega_x\left(t_i - t_j\right) + \theta_i - \theta_j\right]\right\}\sum_{i,j} E\left\{\cos\left(\theta_i - \theta_j\right)\right\} + \\ &+ \sum_{i,j,k,l} E\left\{\cos\left(\theta_i - \theta_j + \theta_k - \theta_l\right)\right\} - \left[\sum_{i,j} E\left\{\cos\left(\theta_i - \theta_j\right)\right\}\right]^2.\end{aligned} \quad (34)$$

The double summation in the 1st and 4th terms of the second member is equal to:

$$\sum_{i,j} E\left\{\cos\left(2\omega_x t_i - 2\omega_x t_j + \theta_i - \theta_j\right)\right\} =$$
$$= \sum_{i,j} \begin{cases} 1 & , i = j \\ e^{-\sigma_\theta^2} \cos\left(2\omega_x t_i - 2\omega_x t_j\right) & , i \neq j \end{cases} =$$
$$= \sum_{i,j} e^{-\sigma_\theta^2} \cos\left(2\omega_x t_i - 2\omega_x t_j\right) - \sum_{i=j} e^{-\sigma_\theta^2} \cos\left(2\omega_x t_i - 2\omega_x t_j\right) + \sum_i 1 = \tag{35}$$
$$= 0 - M e^{-\sigma_\theta^2} + M =$$
$$= M\left(1 - e^{-\sigma_\theta^2}\right).$$

The double summation in the 4th and 6th terms is:

$$\sum_{i,j} E\left\{\cos\left(\theta_i - \theta_j\right)\right\} = \sum_{i,j} \begin{cases} 1 & , i = j \\ e^{-\sigma_\theta^2} & , i \neq j \end{cases} = \sum_{i,j} e^{-\sigma_\theta^2} - \sum_i e^{-\sigma_\theta^2} + \sum_i 1 = M^2 e^{-\sigma_\theta^2} - M e^{-\sigma_\theta^2} + M. \tag{36}$$

The other summations are computed in the Appendixes. Inserting (35), (36), (47), (50) and (53) into (34) leads to:

$$\sigma_{\hat{A}^2}^2 = \frac{A^4}{M}\left(4e^{-\sigma_\theta^2} - 6e^{-2\sigma_\theta^2} + 2e^{-3\sigma_\theta^2}\right) + \frac{A^4}{M^2}\left(4 - 20e^{-\sigma_\theta^2} + 29e^{-2\sigma_\theta^2} - 14e^{-3\sigma_\theta^2} + e^{-4\sigma_\theta^2}\right) + \\ + \frac{A^4}{M^3}\left(-6 + 24e^{-\sigma_\theta^2} - 36e^{-2\sigma_\theta^2} + 24e^{-3\sigma_\theta^2} - 6e^{-4\sigma_\theta^2}\right). \tag{37}$$

5. Bias of the Estimated Sine Wave Amplitude

We are going to use the Taylor series to approximate the non linear relation between square amplitude and amplitude by a polynomial. This allows us to approximately determine the expected value of the estimated amplitude from the expected value of the square amplitude, given by (15), and the variance of the square amplitude, given by (37) as done in [16]:

$$\mu_{\hat{A}} \simeq \sqrt{\mu_{\hat{A}^2}} - \frac{\sigma_{\hat{A}^2}^2}{8\sqrt{\mu_{\hat{A}^2}^3}}. \tag{38}$$

We define now the relative error of the estimation as:

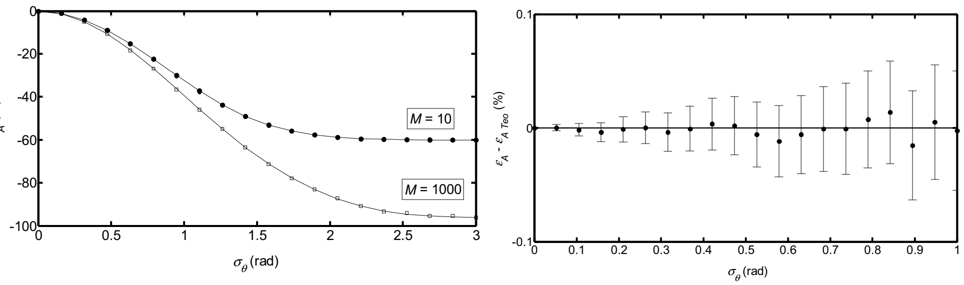
$$\varepsilon_A = \frac{\mu_{\hat{A}} - A}{A}. \tag{39}$$

Inserting (15), (37) into (38) and (38) into (39), leads to:

$$\varepsilon_A \simeq \sqrt{e^{-\sigma_\theta^2} + \frac{2}{M}\left(1 - e^{-\sigma_\theta^2}\right)} - \frac{\begin{bmatrix} \frac{1}{M}\left(4e^{-\sigma_\theta^2} - 6e^{-2\sigma_\theta^2} + 2e^{-3\sigma_\theta^2}\right) + \frac{1}{M^2}\left(4 - 20e^{-\sigma_\theta^2} + 29e^{-2\sigma_\theta^2} - 14e^{-3\sigma_\theta^2} + e^{-4\sigma_\theta^2}\right) + \\ + \frac{1}{M^3}\left(-6 + 24e^{-\sigma_\theta^2} - 36e^{-2\sigma_\theta^2} + 24e^{-3\sigma_\theta^2} - 6e^{-4\sigma_\theta^2}\right) \end{bmatrix}}{8\left[e^{-\sigma_\theta^2} + \frac{2}{M}\left(1 - e^{-\sigma_\theta^2}\right)\right]^{\frac{3}{2}}}, \tag{40}$$

which is the relative bias of the sine wave amplitude estimation using the IEEE 1057 sine fitting algorithm in the presence of jitter. Note that the relative error does not depend on the sine wave amplitude, but only on the number of samples and the phase noise (or jitter) standard deviation.

In order to validate the approximation made in (38) and to check the correctness of the derivations carried out, we did a Monte Carlo [17] analysis of the estimator bias by simulating on a computer a set of data points from a sine wave with sampling instants corrupted by jitter, applying the sine fitting to estimate the amplitude and repeated the procedure 10^4 times to compute the expected value of the estimated amplitude. In Fig. 1a (markers) the relative error obtained is depicted as a function of the phase noise standard deviation for 10 and for 1000 samples (M).



a) b)

Fig. 1. a) Relative error of the estimated sine wave amplitude as a function of phase noise standard deviation (markers, left). A 2 V sine wave with $f_s/f_x = M$ was used and 10^4 repetitions were carried out. The confidence intervals for a confidence level of 99.9 % are too small to be represented graphically. The solid lines represent the theoretical value given by (40). b) Difference between estimated amplitude relative error and theoretical value.

It can be seen that the relative error of the expected value of the estimated amplitude obtained through numerical simulation, is in accordance with the theoretical value given by (40). In Fig. 1b, the deviation of estimated amplitude relative error and theoretical value as a function of phase noise standard deviation is shown with confidence intervals corresponding to 99.9 % confidence level for a normal distribution. All confidence intervals are around 0 (null deviation from numerical simulation and theoretical values) which shows that the approximation made in (38) is valid for the conditions simulated.

From (40) we can compute the limit when the number of samples goes to infinity:

$$\lim_{M \to \infty} \varepsilon_A = e^{\frac{-1}{2}\sigma_\theta^2} - 1. \tag{41}$$

This, which is the result obtained in [15], shows that the estimator is asymptotically biased in the presence of jitter since the relative error does not go to 0 when the number of samples tends to infinity.

6. Conclusions

The expression derived here for the bias of the fitted sine wave amplitude obtained with the 3-parameter sine algorithm, given in (40), shows that the estimator is biased when the acquired samples are affected by jitter which can be due to the analog converter itself or to phase noise in the sampling clock. The existence of this bias was previously mentioned in [15] but only the case of an infinite number of samples was considered. Here we presented an expression that allows the computation of the estimator relative bias given the number of acquired samples and the standard deviation of the jitter or phase noise.

Expression (40) can be used to correct the bias of the estimator if the amount of jitter present is known which can be accomplished using, for instance, the methods recommended in [6].

We limited here our study to the effect of jitter on the estimation of the sine wave amplitude, however we proceed doing work on the effect of jitter on other estimators related to the sine fitting, namely the sine wave offset, initial phase and frequency as well as other parameters derived from them like the module and argument of impedances determined with the help of sine fitting, or signal to noise and distortion ratio (SINAD) of analogue-to-digital converters.

The influence of other non-ideal factors, like harmonic distortion, additive noise and frequency error, on the bias and on the variance of the estimators has also to be studied in the future to achieve a full understanding of the performance of sine fitting algorithms in real conditions.

Appendix A

In this appendix we compute the first term in the second member of (34). To determine an expression for the expected value we have to consider whether some of the indices are equal because is such cases the random variables θ will cancel each other out. There are 14 different cases where one or more of the 4 indices i, j, k and l are equal. Those cases are illustrated in (42). To make it easier to read the expressions that follow, we have chosen to attribute different symbols $(\bullet \times \circ *)$ to the indices. For example, the first case in (42) is identified by the symbols $(\bullet\bullet\bullet\bullet)$. This means that all the 4 indices are the same. Note that this case encompasses many different possible values of the indices (they can be all equal to 1 or 2, or 3, *etc.*). In the second case in (42), for example, indices i, j and k are the same and index l is different $(\bullet\bullet\bullet\times)$. In the last case in (42) all the 4 indices are different.

$$\sum_{i,j,k,l} E\left\{\cos\left[\omega_x\left(t_i - t_j + t_k - t_l\right) + \left(\theta_i - \theta_j + \theta_k - \theta_l\right)\right]\right\} = \sum_{i,j,k,l} \begin{cases} 1 & \bullet\bullet\bullet\bullet & M \\ e^{-\sigma_\theta^2}\cos\left(\omega_x t_k - \omega_x t_l\right) & \bullet\bullet\bullet\times & -M \\ e^{-\sigma_\theta^2}\cos\left(\omega_x t_k - \omega_x t_l\right) & \bullet\bullet\times\bullet & -M \\ e^{-\sigma_\theta^2}\cos\left(\omega_x t_i - \omega_x t_j\right) & \bullet\times\bullet\bullet & -M \\ e^{-\sigma_\theta^2}\cos\left(\omega_x t_i - \omega_x t_j\right) & \times\bullet\bullet\bullet & -M \\ 1 & \bullet\bullet\times\times & M(M-1) \\ e^{-4\sigma_\theta^2}\cos\left(2\omega_x t_i - 2\omega_x t_j\right) & \bullet\times\bullet\times & -M \\ 1 & \bullet\times\times\bullet & M(M-1) \\ e^{-\sigma_\theta^2}\cos\left(\omega_x t_k - \omega_x t_l\right) & \bullet\bullet\times\circ & -M(M-2) \\ e^{-3\sigma_\theta^2}\cos\left(2\omega_x t_i - \omega_x t_j - \omega_x t_l\right) & \bullet\times\bullet\circ & 2M \\ e^{-\sigma_\theta^2}\cos\left(\omega_x t_j - \omega_x t_k\right) & \bullet\times\circ\bullet & -M(M-2) \\ e^{-\sigma_\theta^2}\cos\left(\omega_x t_i - \omega_x t_l\right) & \times\bullet\bullet\circ & -M(M-2) \\ e^{-3\sigma_\theta^2}\cos\left(\omega_x t_i - 2\omega_x t_j + \omega_x t_k\right) & \times\bullet\circ\bullet & 2M \\ e^{-\sigma_\theta^2}\cos\left(\omega_x t_i - \omega_x t_j\right) & \times\circ\bullet\bullet & -M(M-2) \\ e^{-2\sigma_\theta^2}\cos\left(\omega_x t_i - \omega_x t_j + \omega_x t_k - \omega_x t_l\right) & \bullet\times\circ * & 2M^2 - 6M \end{cases} \quad (42)$$

Each of the 15 cases in the curly bracket corresponds to a different set of values of i, j, k and l. Since all those cases are mutually exclusive, the quadruple summation of the bracket in (42) can be expressed as the sum of 15 summations with different arguments (the ones in the curly bracket). The value of those summations, not considering the exponential term, is indicated in the extreme-right column of (42).

We will look now at how the value of some of those summations was obtained. The first summation in the curly brackets is the number of elements which in this case is M. The second summation can be seen as a complete double summation minus the cases where $k = l$:

$$\sum_{k\neq l}\cos\left(\omega_x t_k-\omega_x t_l\right)=\sum_{k,l}\cos\left(\omega_x t_k-\omega_x t_l\right)-\sum_{k=l}\cos\left(\omega_x t_k-\omega_x t_l\right)=0-M=-M\,. \qquad (43)$$

The complete summation has a null value since we have an integer number of periods of the cosine function and there are M cases where $k = l$. The summation will thus be – M.
The sixth summation is:

$$\sum_{i\neq j}1=\sum_{i,j}1-\sum_{i=j}1=M^2-M=M\left(M-1\right). \qquad (44)$$

The ninth summation is:

$$\sum_{j\neq k\neq l}\cos\left(\omega_x t_k-\omega_x t_l\right)=\left(M-2\right)\sum_{k\neq l}\cos\left(\omega_x t_k-\omega_x t_l\right)=-M\left(M-2\right). \qquad (45)$$

The argument of this summation does not depend on j and there are M – 2 values of j which are different form k and l. This term thus has M – 2 times the summation on k and l which, as was seen in (43), equals – M.

The 13th summation is a triple summation which can be split into a complete triple summation minus the cases where two or three indices are the same. The complete summation is null so we have:

$$\begin{aligned}\sum_{i\neq j\neq k}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]=&-\sum_{i=j\neq k}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]-\sum_{i\neq j=k}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]-\\&-\sum_{i=k\neq j}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]-\sum_{i=j=k}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]\end{aligned}\,. \qquad (46)$$

The first 3 summations in the second member of (46) are equal to – M. The argument of the cosine in last summation is null since all the indices are the same. As the indices go from 1 to M, there are M terms equal to 1 (cos(0)). The last term in (46) is thus M. The 13th summation in (42) is thus equal to 2M.

The last summation in (42) has a quadruple summation where all the indices have different values. This partial summation can be seen as a complete quadruple summation minus the cases where some or all the indices are equal. The complete summation has a null value since we have an integer number of periods of the cosine function and there are $6M-2M^2$ cases where some or all the indices are equal. This is just the sum of the cases in all the other terms.

Putting all the terms together leads to:

$$\begin{aligned}\sum_{i,j,k,l}E\left\{\cos\left[\omega_x\left(t_i-t_j+t_k-t_l\right)+\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}=M+2M\left(M-1\right)-4Me^{-\sigma_\theta^2}-Me^{-4\sigma_\theta^2}-4M\left(M-2\right)e^{-\sigma_\theta^2}+\\+4Me^{-3\sigma_\theta^2}+\left(2M^2-6M\right)e^{-2\sigma_\theta^2}=M\left(-1+4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+4e^{-3\sigma_\theta^2}-e^{-4\sigma_\theta^2}\right)+2M^2\left(1-2e^{-\sigma_\theta^2}+e^{-2\sigma_\theta^2}\right).\end{aligned} \qquad (47)$$

Appendix B

In this appendix we compute the third term in the second member of (34). Here we proceed as we did in Appendix B. All the 15 cases where the 4 indices can be equal to each other are enumerated and the expected value is computed individually for each of those cases.

$$\sum_{i,j,k,l} E\left\{\cos\left[\omega_x\left(t_i-t_j\right)+\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}=\sum_{i,j,k,l}\begin{cases}1 & \bullet\bullet\bullet\bullet \quad M\\ e^{-\sigma_\theta^2} & \bullet\bullet\bullet\times \quad M(M-1)\\ e^{-\sigma_\theta^2} & \bullet\bullet\times\bullet \quad M(M-1)\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\bullet\bullet \quad -M\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \times\bullet\bullet\bullet \quad -M\\ 1 & \bullet\bullet\times\times \quad M(M-1)\\ e^{-4\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\bullet\times \quad -M\\ \cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\times\bullet \quad -M\\ e^{-\sigma_\theta^2} & \bullet\bullet\times\circ \quad M(M-1)(M-2)\\ e^{-3\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\bullet\circ \quad -M(M-2)\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_j-\omega_x t_j\right] & \bullet\times\circ\bullet \quad -M(M-2)\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \times\bullet\bullet\circ \quad -M(M-2)\\ e^{-3\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \times\bullet\circ\bullet \quad -M(M-2)\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \times\circ\bullet\bullet \quad -M(M-2)\\ e^{-2\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\circ* \quad -M(M-2)(M-3)\end{cases} \tag{48}$$

Again, in the right extreme-right of (34) we indicate the value of the summations without considering the exponential terms. Equation (48) thus becomes:

$$\begin{aligned}\sum_{i,j,k,l} E\left\{\cos\left[\omega_x\left(t_i-t_j\right)+\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}&=\left[-M(M-2)(M-3)\right]e^{-2\sigma_\theta^2}+\left[-2M(M-2)\right]e^{-3\sigma_\theta^2}-Me^{-4\sigma_\theta^2}+\\ &+\left[M+M(M-1)-M\right]+\left[2M(M-1)-2M+M(M-1)(M-2)-3M(M-2)\right]e^{-\sigma_\theta^2}=\\ &=\left[M^2-M\right]+\left[4M-4M^2+M^3\right]e^{-\sigma_\theta^2}+\left[-6M+5M^2-M^3\right]e^{-2\sigma_\theta^2}+\left[4M-2M^2\right]e^{-3\sigma_\theta^2}-Me^{-4\sigma_\theta^2}\end{aligned} \tag{49}$$

This can be further simplified to:

$$\begin{aligned}&\sum_{i,j,k,l} E\left\{\cos\left[\omega_x\left(t_i-t_j\right)+\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}=\\ &=M\left(-1+4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+4e^{-3\sigma_\theta^2}-e^{-4\sigma_\theta^2}\right)+M^2\left(1-4e^{-\sigma_\theta^2}+5e^{-2\sigma_\theta^2}-2e^{-3\sigma_\theta^2}\right)+M^3\left(e^{-\sigma_\theta^2}-e^{-2\sigma_\theta^2}\right).\end{aligned} \tag{50}$$

Appendix C

In this appendix we compute the fifth term in the second member of (34) as was done in Appendix B and C. The 15 different cases are:

$$\sum_{i,j,k,l} E\left\{\cos\left[\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}=\sum_{i,j,k,l}\begin{cases} 1 & \bullet\bullet\bullet\bullet & M \\ e^{-\sigma_\theta^2} & \bullet\bullet\bullet\times & M(M-1) \\ e^{-\sigma_\theta^2} & \bullet\bullet\times\bullet & M(M-1) \\ e^{-\sigma_\theta^2} & \bullet\times\bullet\bullet & M(M-1) \\ e^{-\sigma_\theta^2} & \times\bullet\bullet\bullet & M(M-1) \\ 1 & \bullet\bullet\times\times & M(M-1) \\ e^{-4\sigma_\theta^2} & \bullet\times\bullet\times & M(M-1) \\ 1 & \bullet\times\times\bullet & M(M-1) \\ e^{-\sigma_\theta^2} & \bullet\bullet\times\circ & M(M-1)(M-2) \\ e^{-3\sigma_\theta^2} & \bullet\times\bullet\circ & M(M-1)(M-2) \\ e^{-\sigma_\theta^2} & \bullet\times\circ\bullet & M(M-1)(M-2) \\ e^{-\sigma_\theta^2} & \times\bullet\bullet\circ & M(M-1)(M-2) \\ e^{-3\sigma_\theta^2} & \times\bullet\circ\bullet & M(M-1)(M-2) \\ e^{-\sigma_\theta^2} & \times\circ\bullet\bullet & M(M-1)(M-2) \\ e^{-2\sigma_\theta^2} & \bullet\times\circ * & M(M-1)(M-2)(M-3) \end{cases} \tag{51}$$

This summation thus becomes:

$$\begin{aligned} &\sum_{i,j,k,l} E\left\{\cos\left[\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}= \\ &\quad=\left[M+2M(M-1)\right]+\left[4M(M-1)+4M(M-1)(M-2)\right]e^{-\sigma_\theta^2}\left[M(M-1)(M-2)(M-3)\right]e^{-2\sigma_\theta^2}+ \\ &\quad+\left[2M(M-1)(M-2)\right]e^{-3\sigma_\theta^2}+\left[M(M-1)\right]e^{-4\sigma_\theta^2}= \\ &\quad=\left[-M+2M^2\right]+\left[4M(M-1)(M-1)\right]e^{-\sigma_\theta^2}+\left[M(M-1)(M-2)(M-3)\right]e^{-2\sigma_\theta^2}+ \\ &\quad+\left[2M(M-1)(M-2)\right]e^{-3\sigma_\theta^2}+\left[M(M-1)\right]e^{-4\sigma_\theta^2}. \end{aligned} \tag{52}$$

Simplifying leads to:

$$\begin{aligned} \sum_{i,j,k,l} E\left\{\cos\left[\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}&=M\left(-1+4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+4e^{-3\sigma_\theta^2}-e^{-4\sigma_\theta^2}\right)+ \\ &+M^2\left(2-8e^{-\sigma_\theta^2}+11e^{-2\sigma_\theta^2}-6e^{-3\sigma_\theta^2}+e^{-4\sigma_\theta^2}\right)+M^3\left(4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+2e^{-3\sigma_\theta^2}\right)+M^4\left(e^{-2\sigma_\theta^2}\right). \end{aligned} \tag{53}$$

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